

The asymptotic Schottky problem

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Abstract

Let \mathcal{M}_g denote the moduli space of compact Riemann surfaces of genus g and let \mathcal{A}_g be the space of principally polarized abelian varieties of (complex) dimension g . Let $J : \mathcal{M}_g \rightarrow \mathcal{A}_g$ be the map which associates to a Riemann surface its Jacobian. The map J is injective, and the image $J(\mathcal{M}_g)$ is contained in a proper subvariety of \mathcal{A}_g when $g \geq 4$. The classical and long-studied Schottky problem is to characterize the Jacobian locus $\mathcal{J}_g := J(\mathcal{M}_g)$ in \mathcal{A}_g . In this paper we address a large scale version of this problem posed by Farb and called the *coarse Schottky problem*: How does \mathcal{J}_g look “from far away”, or how “dense” is \mathcal{J}_g in the sense of coarse geometry? The coarse geometry of the Siegel modular variety \mathcal{A}_g is encoded in its asymptotic cone $\text{Cone}_\infty(\mathcal{A}_g)$, which is a Euclidean simplicial cone of (real) dimension g . Our main result asserts that the Jacobian locus \mathcal{J}_g is “asymptotically large”, or “coarsely dense” in \mathcal{A}_g . More precisely, the subset of $\text{Cone}_\infty(\mathcal{A}_g)$ determined by \mathcal{J}_g actually coincides with this cone. The proof also shows that the Jacobian locus of hyperelliptic curves is coarsely dense in \mathcal{A}_g as well. We also study the boundary points of the Jacobian locus \mathcal{J}_g in \mathcal{A}_g and in the Baily-Borel and the Borel-Serre compactification. We show that for large genus g the set of boundary points of \mathcal{J}_g in these compactifications is “small”.

1 Introduction

The *Siegel upper half space* \mathcal{H}_g is a Hermitian symmetric space of noncompact type which generalizes the Poincaré upper half plane:

$$\mathcal{H}_g := \{Z \in \mathbb{C}^{g \times g} \mid Z \text{ symmetric, } \text{Im } Z \text{ positive definite}\}.$$

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The symplectic group

$$Sp(g, \mathbb{R}) = \{x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2g, \mathbb{R}) \mid A, B, C, D \in M_n(\mathbb{R}), x^t J_g x = J_g\},$$

where $J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ and I_g is the identity $g \times g$ matrix, acts isometrically, holomorphically and transitively on \mathcal{H}_g by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z := (AZ + B)(CZ + D)^{-1}.$$

As the stabilizer of the point $iI_g \in \mathcal{H}_g$ is isomorphic to $U(g)$, one has the identification $\mathcal{H}_g \cong Sp(g, \mathbb{R})/U(g)$.

The *Siegel modular group* $Sp(g, \mathbb{Z})$ is an arithmetic subgroup of $Sp(g, \mathbb{R})$ which acts properly discontinuously on \mathcal{H}_g . The corresponding quotient, $\mathcal{A}_g := Sp(g, \mathbb{Z}) \backslash \mathcal{H}_g$ is called the *Siegel modular variety*, and can be identified with the moduli space of principally polarized abelian varieties (or tori) of complex dimension g . By [1], \mathcal{A}_g is a quasi-projective variety and admits a compactification that is a normal projective variety. In the following, this compactification is called the Baily-Borel compactification and denoted by $\overline{\mathcal{A}}_g^{BB}$, in view of the corresponding compactification for general arithmetic Hermitian locally symmetric spaces constructed in [3].

The *moduli space* \mathcal{M}_g of compact Riemann surfaces of genus $g > 0$ is a complex (Kähler) manifold (or rather orbifold) of dimension $3g - 3$. Let $M \in \mathcal{M}_g$ be a Riemann surface and consider a symplectic basis $\{A_j, B_j\}$ for the first homology group $H_1(M, \mathbb{Z})$ of M . Associated to this basis is a normalized basis $\{\omega_1, \dots, \omega_g\}$ of holomorphic 1-forms (or abelian differentials of the first kind) satisfying $\int_{A_k} \omega_l = \delta_{kl}$. The corresponding *period matrix* Π of M is the complex $g \times g$ matrix with entries $\Pi_{ij} := \int_{B_i} \omega_j$. Riemann's bilinear relations [14, p. 232] are equivalent to that $\Pi = (\Pi_{ij})$ belongs to the Siegel upper half space \mathcal{H}_g . Then $L := \mathbb{Z}^g \oplus \Pi \cdot \mathbb{Z}^g$ is a lattice in \mathbb{C}^g and the *Jacobian* of the Riemann surface M is the torus \mathbb{C}^g/L , which turns out to be an abelian variety, i.e., it admits the structure of a projective variety. Moreover, the intersection pairing on homology $H_1(M, \mathbb{Z})$ determines a Hermitian bilinear form on \mathbb{C}^g with respect to which the torus \mathbb{C}^g/L is principally polarized [14, p. 359].

The choice of a different homology basis of $H_1(M, \mathbb{Z})$ yields a matrix $\Pi' = \gamma \cdot \Pi$ for some $\gamma \in Sp(g, \mathbb{Z})$ and hence a Jacobian in the same isomorphy class of principally polarized abelian varieties. We thus have the well-defined *Jacobian (or period) map*

$$J : \mathcal{M}_g \longrightarrow \mathcal{A}_g$$

which associates to a Riemann surface M its Jacobian $J(M)$. Intrinsically, the Jacobian variety $J(M)$ is equal to $(H^0(M, \Omega^1))^*/H_1(M, \mathbb{Z})$, where $H^0(M, \Omega^1)$ is the space of holomorphic 1-forms, and the inclusion of $H_1(M, \mathbb{Z})$ in the dual space $(H^0(M, \Omega^1))^*$ is obtained by integrating 1-forms along cycles in $H_1(M, \mathbb{Z})$ [14, p. 36]. By Torelli's Theorem (see [14, p. 359]), the Jacobian map J is injective. The classical *Schottky problem* is to characterize the *Jacobian (or period) locus* $\mathcal{J}_g := J(\mathcal{M}_g)$ inside the space \mathcal{A}_g of all principally polarized abelian varieties.

A lot of work has been done on this important problem. Basically there are two kind of approaches: (1) the analytic approach, finding equations that “cut out” the locus $J(\mathcal{M}_g)$ inside \mathcal{A}_g ; (2) the geometric approach, finding geometric properties of a principally polarized abelian variety that are satisfied only by Jacobians. For an nice discussion of the Schottky problem, see [25]. More recent surveys of the status of the Schottky problem are [4] and [10].

In [9] Buser and Sarnak studied the position of the Jacobian locus \mathcal{J}_g in \mathcal{A}_g for large genera g . They consider a certain (systolic) function m which can be thought of as giving a “distance” to the boundary of \mathcal{A}_g . Then they prove that

$$\mathcal{J}_g \subset N_g := \{x \in \mathcal{A}_g \mid m(x) \leq \frac{3}{\pi} \log(4g + 3)\}.$$

Moreover, as $g \rightarrow +\infty$, $\text{Vol}(N_g)/\text{Vol}(\mathcal{A}_g) = O(g^{-\nu g})$ for any $\nu < 1$. This means that for large genus the entire Jacobian locus lies in a very small neighbourhood N_g of the boundary of \mathcal{A}_g .

Motivated by this work of Buser and Sarnak, B. Farb proposed in [13, Problem 4.11] to study the Schottky problem from the point of view of large scale geometry, called the “Coarse Schottky Problem”: How does \mathcal{J}_g look “from far away”, or how “dense” is \mathcal{J}_g inside \mathcal{A}_g in the sense of coarse geometry?

This question can be made precise by using the concept of an *asymptotic cone* (or *tangent cone at infinity*) introduced by Gromov. Recall that a sequence (X_n, p_n, d_n) of unbounded, pointed metric spaces converges in the sense of Gromov-Hausdorff, or *Gromov-Hausdorff-converges*, to a pointed metric space (X, p, d) if for every $r > 0$, the Hausdorff-distance between the balls $B_r(p_n)$ in (X_n, d_n) and the ball $B_r(p)$ in (X, d) goes to zero as $n \rightarrow \infty$ (see [15], Chapter 3). Let x_0 be an (arbitrary) point of \mathcal{A}_g . The *asymptotic cone* of \mathcal{A}_g endowed with the locally symmetric metric $d_{\mathcal{A}_g}$ is defined as the Gromov-Hausdorff-limit of rescaled pointed spaces:

$$\text{Cone}_\infty(\mathcal{A}_g) := \mathcal{GH} - \lim_{n \rightarrow \infty} (\mathcal{A}_g, x_0, \frac{1}{n} d_{\mathcal{A}_g}).$$

Note that $\text{Cone}_\infty(\mathcal{A}_g)$ is independent of the choice of the base point x_0 .

We remark that in contrast to the case considered here, the definition of an asymptotic cone in general involves the use of ultrafilters, and the limit space may depend on the chosen ultrafilter. Various aspects of asymptotic cones of general spaces are discussed in Gromov's book [15] (see also [22]). In some cases asymptotic cones are easy to describe. For example, the asymptotic cone of the Euclidean space \mathbb{R}^n is isometric to \mathbb{R}^n . Similarly, if C is a cone in \mathbb{R}^n , then $\text{Cone}_\infty(C)$ is isometric to C . For another class of examples, let V be a finite volume Riemannian manifold of strictly negative sectional curvature and with k cusps, in particular V may be a non-compact, finite volume quotient of a rank 1 symmetric space. Then $\text{Cone}_\infty(V)$ is a “cone” over k points, i.e., k rays with a common origin. For Siegel's modular variety, $\text{Cone}_\infty(\mathcal{A}_g)$ is known to be isometric to a g -dimensional metric cone over a simplex (see Section 2 below).

Farb's question can now be stated as follows [13, Problem 4.11]:

Coarse Schottky problem: *Describe, as a subset of a g -dimensional metric cone, the subset of $\text{Cone}_\infty(\mathcal{A}_g)$ determined by the Jacobian locus \mathcal{J}_g in \mathcal{A}_g .*

Farb also asked to determine the metric distortion of \mathcal{J}_g inside \mathcal{A}_g [13, Problem 4.12]. See §8 below for some comments on that problem.

Our first result solves the coarse Schottky problem. It asserts that the locus \mathcal{J}_g is asymptotically “dense”. More precisely, we have

Theorem 1.1 *Let $\text{Cone}_\infty(\mathcal{A}_g)$ be the asymptotic cone of Siegel's modular variety. Then the subset of $\text{Cone}_\infty(\mathcal{A}_g)$ determined by the Jacobian locus $\mathcal{J}_g \subset \mathcal{A}_g$ is equal to the entire $\text{Cone}_\infty(\mathcal{A}_g)$. More specifically, there exists a constant δ_g depending only on g such that \mathcal{A}_g is contained in a δ_g -neighbourhood of \mathcal{J}_g .*

In view of the results of Buser and Sarnak Theorem 1.1 might be surprising at first sight. Note however that [9] deals with the asymptotic situation when the genus $g \rightarrow \infty$, while the genus g is fixed in the present paper. The result of Buser and Sarnak implies that the constant $\delta_g \rightarrow \infty$. A open problem is to find an effective bound on δ_g .

Hyperelliptic curves are special among curves and have been intensively studied in algebraic geometry. When the genus g is at least 3, a generic curve in \mathcal{M}_g is not hyperelliptic. In fact, denote the subspace of \mathcal{M}_g consisting of hyperelliptic curves by \mathcal{HE}_g . Then $\dim \mathcal{HE}_g = 2g - 1$ (see [14, pp. 255-256]). Since $\dim \mathcal{M}_g = 3g - 3$, \mathcal{HE}_g is a proper subvariety of \mathcal{M}_g when $g \geq 3$. Again one can ask about the coarse density of the image $J(\mathcal{HE}_g)$ in \mathcal{A}_g . The answer is

Theorem 1.2 *The subset of $\text{Cone}_\infty(\mathcal{A}_g)$ determined by the hyperelliptic Jacobian locus $J(\mathcal{HE}_g) \subset \mathcal{A}_g$ is equal to the entire $\text{Cone}_\infty(\mathcal{A}_g)$. Also, there exists a constant δ'_g depending only on g such that \mathcal{A}_g is contained in the δ'_g -neighborhood of \mathcal{HE}_g .*

Siegel's modular variety \mathcal{A}_g is an arithmetic Hermitian locally symmetric space and thus admits several compactifications, which are motivated by various applications (see e.g. [7]). The compactification $\overline{\mathcal{A}}_g^{BB}$ mentioned above is a special case of the Baily-Borel compactification which exists for general arithmetic Hermitian locally symmetric spaces (see [3]). The Baily-Borel compactification is a normal projective variety. We denote its boundary $\overline{\mathcal{A}}_g^{BB} - \mathcal{A}_g$ by $\partial\overline{\mathcal{A}}_g^{BB}$.

There is another, larger compactification of arithmetic locally symmetric spaces $\Gamma \backslash X$ constructed in [8], called the Borel-Serre compactification and denoted by $\overline{\Gamma \backslash X}^{BS}$. It is a manifold with corners and the inclusion $\Gamma \backslash X \hookrightarrow \overline{\Gamma \backslash X}^{BS}$ is a homotopy equivalence when Γ is torsion-free. This Borel-Serre compactification has many important applications in topology. The basic reason is that $\overline{\Gamma \backslash X}^{BS}$ is a classifying space of Γ which has the structure of a finite CW-complex and the topology of its boundary can be described by the rational Tits building of the associated algebraic group. We denote the Borel-Serre compactification of \mathcal{A}_g by $\overline{\mathcal{A}}_g^{BS}$ and its boundary by $\partial\overline{\mathcal{A}}_g^{BS}$.

Since each compactification of \mathcal{A}_g reflects certain structures or sizes “near infinity”, it is natural to consider the boundary points of the period locus \mathcal{J}_g in these two compactifications. Let $\overline{\mathcal{J}}_g^{BB}$ be the closure of \mathcal{J}_g in $\overline{\mathcal{A}}_g^{BB}$, and $\partial\overline{\mathcal{J}}_g^{BB} = \overline{\mathcal{J}}_g^{BB} \cap \partial\overline{\mathcal{A}}_g^{BB}$ be the boundary of $\overline{\mathcal{J}}_g^{BB}$ in $\overline{\mathcal{A}}_g^{BB}$. Similarly, let $\partial\overline{\mathcal{J}}_g^{BS}$ be the boundary of $\overline{\mathcal{J}}_g^{BS}$ in $\overline{\mathcal{A}}_g^{BS}$.

Our next results show that these boundaries form “small” proper subsets when g is large.

Theorem 1.3 *When $g = 2, 3, 4$, the boundary $\partial\overline{\mathcal{J}}_g^{BB}$ is equal to the whole boundary $\partial\overline{\mathcal{A}}_g^{BB}$ of the Baily-Borel compactification. For $g \geq 5$, $\partial\overline{\mathcal{J}}_g^{BB}$ is a proper subvariety of $\partial\overline{\mathcal{A}}_g^{BB}$. In fact, it is the union of the Jacobian loci \mathcal{J}_k of Riemann surfaces of lower genus k , $k \leq g - 1$.*

Corollary 1.4 *When $g = 2, 3$, the boundary $\partial\overline{\mathcal{J}}_g^{BS}$ is equal to the whole boundary $\partial\overline{\mathcal{A}}_g^{BS}$. For $g \geq 5$, $\partial\overline{\mathcal{J}}_g^{BS}$ is a proper subspace of $\partial\overline{\mathcal{A}}_g^{BS}$ of strictly smaller dimension.*

Recall that moduli space \mathcal{M}_g is not compact since there are sequences of compact Riemann surfaces which degenerate (compare Section 3). The Deligne-Mumford compactification $\overline{\mathcal{M}}_g^{DM}$ is a (projective) compactification which is obtained by adding stable Riemann surfaces (see [11]). In proving Theorem 1.3 and Corollary 1.4 we will use the following Proposition 1.5 (which may be of independent interest).

Proposition 1.5 *The Jacobian map $J : \mathcal{M}_g \rightarrow \mathcal{A}_g$ extends to a holomorphic and hence algebraic map $\overline{J} : \overline{\mathcal{M}}_g^{DM} \rightarrow \overline{\mathcal{A}}_g^{BB}$.*

Proposition 1.5 also yields another proof of the result in [2] that the closure of $J(\mathcal{M}_g)$ in $\overline{\mathcal{A}_g}^{BB}$ is a subvariety. In fact, that closure is equal to the image $J(\overline{\mathcal{M}_g}^{DM})$ of a projective variety under an algebraic map. Another consequence of Proposition 1.5 is a precise description of the topological boundary of \mathcal{J}_g in \mathcal{A}_g (see Proposition 7.2 below). For $g = 2, 3$ this in turn determines the complement of \mathcal{J}_g in \mathcal{A}_g (see Corollary 7.4).

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2 The coarse geometry of Siegel modular varieties

Coarse fundamental domains, which are usually called fundamental sets, for arithmetic groups of semisimple Lie groups acting on symmetric spaces of noncompact type are provided by reduction theory (see e.g. [5]).

In order to describe these fundamental sets in the special case of $Sp(g, \mathbb{Z})$ acting on the Siegel upper half space \mathcal{H}_g , we first introduce certain subgroups of $Sp(g, \mathbb{R})$. We set

$$A_g := \left\{ \begin{pmatrix} H & 0 \\ 0 & H^{-1} \end{pmatrix} \in Sp(g, \mathbb{R}) \mid H \text{ positive diagonal} \right\} \text{ and}$$

$$N_g := \left\{ \begin{pmatrix} A & B \\ 0 & {}^t A^{-1} \end{pmatrix} \in Sp(g, \mathbb{R}) \mid A \text{ upper triangular, } 1 \text{ on diagonal; } A {}^t B = B {}^t A \right\}.$$

For $a \in \mathbb{R}_{>0}$ we define the *Weyl chamber* $C_a \subset A_g$ as the subset of those

$$\begin{pmatrix} H & 0 \\ 0 & H^{-1} \end{pmatrix} \in A_g, \quad H = \text{Diag}(h_1, h_2, \dots, h_g)$$

which satisfy the inequalities

$$\begin{aligned} h_i h_j^{-1} &\geq a & \text{for } 1 \leq i < j \leq n \\ h_i h_j &\geq a & \text{for } 1 \leq i \leq j \leq n. \end{aligned}$$

Then, for $\omega \subset N_g$ bounded, a *Siegel set* in \mathcal{H}_g is of the form

$$\mathcal{S}_{a,\omega} := \omega C_a \cdot iI_g,$$

where $I_g \in M_g(\mathbb{C})$ is the identity $g \times g$ matrix and iI_g is the chosen base point of \mathcal{H}_g . Note that $A_g \cdot iI_g$ endowed with the metric induced from \mathcal{H}_g is a maximal totally

geodesic flat submanifold of the symmetric space \mathcal{H}_g and that the Weyl chambers $\mathcal{C}_a = C_a \cdot iI_g \subset A_g \cdot iI_g, a > 0$, are Euclidean cones over a simplex.

The following proposition is a concise version of reduction theory for $Sp(g, \mathbb{Z})$; for a proof see [28] (or also [5]).

Proposition 2.1 *There are $a > 0$ and $\omega \subset N_g$ as above such that $\mathcal{S}_{a,\omega}$ is a fundamental set for $Sp(g, \mathbb{Z})$, i.e.*

- (1) $\mathcal{H}_g = Sp(g, \mathbb{Z}) \cdot \mathcal{S}_{a,\omega}$ and
- (2) the set $\{\gamma \in Sp(g, \mathbb{Z}) \mid \gamma \cdot \mathcal{S}_{a,\omega} \cap \mathcal{S}_{a,\omega} \neq \emptyset\}$ is finite.

We next introduce some additional concepts. A subset \mathcal{N} of a metric space (X, d) is called a (δ) -net if there is a positive constant δ such that $d(p, \mathcal{N}) \leq \delta$ for all $p \in X$; in particular the Hausdorff-distance between \mathcal{N} and X is at most δ . A map between metric spaces $f : (X, d_X) \rightarrow (Y, d_Y)$ is a *quasi-isometric embedding* if there are constants $C \geq 1$ and $D \geq 0$ such that for all $p, q \in X$ one has

$$C^{-1}d_X(x_1, x_2) - D \leq d_Y(f(x_1), f(x_2)) \leq Cd_X(x_1, x_2) + D.$$

If, in addition, the image $f(X)$ is a δ -net in Y for some $\delta > 0$, then f is a *quasi-isometry*.

Now let $\pi : \mathcal{H}_g \rightarrow \mathcal{A}_g = Sp(g, \mathbb{Z}) \backslash \mathcal{H}_g$ denote the canonical projection and let \mathcal{A}_g be endowed with the locally symmetric metric such that π is a Riemannian covering. Proposition 2.1 yields that $\mathcal{A}_g = \pi(\mathcal{S}_{a,\omega})$ and that π is a uniformly bounded finite-to-one map. Furthermore the following metric properties hold.

Proposition 2.2 *Let $\mathcal{S}_{a,\omega}$ be a Siegel set as in Proposition 2.1, then there is $a^* \geq a$ such that π restricted to the Weyl chamber $\mathcal{C}_{a^*} = C_{a^*} \cdot iI_g \subset \mathcal{S}_{a,\omega}$ is an isometry. Moreover, $\pi(\mathcal{C}_{a^*})$ is a net in \mathcal{A}_g . In particular, Siegel's modular variety \mathcal{A}_g is quasi-isometric to the Euclidean cone $\mathcal{C}_{a^*} \cong \pi(\mathcal{C}_{a^*})$ with the multiplicative constant in the quasi-isometry equal to 1.*

For a proof (in the general setting of locally symmetric spaces) see [23, Theorem 4.1, Corollary 4.2] and [21, Lemma 5.10].

The asymptotic cones of general locally symmetric spaces of higher rank have been determined by Hattori, Ji-MacPherson and Leuzinger (see [19], [21], [23]). Since the asymptotic cone of a Euclidean cone over a simplex is equal to itself, we have the following identification for Siegel's modular variety.

Proposition 2.3 *Let \mathcal{C}_{a^*} be as in Proposition 2.2 and let $\pi(\mathcal{C}_{a^*})$ its isometric image in \mathcal{A}_g . Then the asymptotic cone, $\text{Cone}_\infty(\mathcal{A}_g)$, of \mathcal{A}_g is isometric to the Euclidean cone $\pi(\mathcal{C}_{a^*})$, which is a Euclidean cone over a simplex.*

Remark. The general result proved in [19], [21], [23] is as follows. Let $V = \Gamma \backslash G/K$ be a locally symmetric space. Then $\text{Cone}_\infty(V)$ is isometric to the Euclidean cone over a finite simplicial complex $\Gamma \backslash \Delta_\mathbb{Q}$, the quotient by Γ of the rational Tits building $\Delta_\mathbb{Q}$ of G . In the special case of Siegel modular varieties the quotient of the Tits building is just *one* simplex, and the Euclidean cone over it is isometric to the positive Weyl chamber \mathcal{C}_{a^*} . This corresponds to the fact that there is only one $Sp(g, \mathbb{Z})$ -conjugacy class of minimal \mathbb{Q} -parabolic subgroups of $G = Sp(n, \mathbb{R})$. Thus $\text{Cone}_\infty(\mathcal{A}_g)$ is isometric to a Euclidean cone \mathcal{C}_{a^*} .

3 Degenerations of surfaces and period matrices

It is crucial for our approach to obtain information about the image of the Jacobian map $J : \mathcal{M}_g \longrightarrow \mathcal{A}_g$ when restricted to certain “thin parts” of moduli space \mathcal{A}_g , i.e., subsets of \mathcal{M}_g consisting of Riemann surfaces (endowed with a hyperbolic metric) which contain at least one closed geodesic of length less than some fixed small number (see [24] for a precise description of these sets). The basic philosophy is that while it seems to be difficult to describe the points in the Jacobian locus \mathcal{J}_g completely, we can describe certain points in its boundary, and these boundary points in turn allow us to describe the asymptotic cone of \mathcal{J}_g in \mathcal{A}_g . Equivalently, we want to understand the extended Jacobian map $J : \overline{\mathcal{M}}_g^{DM} \rightarrow \overline{\mathcal{A}}_g^{BB}$ from the Deligne-Mumford compactification of \mathcal{M}_g in Proposition 1.5 and the intersection of $J(\overline{\mathcal{M}}_g^{DM})$ with \mathcal{A}_g .

To this end we discuss in this section the degeneration of Riemann surfaces to singular surfaces with nodes. The singular surfaces may be regarded as the union of finitely many compact surfaces with punctures (the latter identified by the local equation $zw = 0$). There are two cases of degeneration in \mathcal{M}_g depending upon whether the node separates the (singular) surface or not. It is well-known that these two types of degeneration yield completely different limiting behaviour in the period locus $\mathcal{J}_g = J(\mathcal{M}_g) \subset \mathcal{A}_g$.

We first discuss a model for the degeneration of M into two surfaces M_1, M_2 with genera $g_1, g_2 > 0$ and joined at a node p . We choose points $p_1, p_2 \in M_1, M_2$ and coordinates $z_i : U_i \rightarrow D$ centered at p_i for $i = 1, 2$ and D the unit disc in \mathbb{C} .

Let $S := \{(z, w, t) \mid zw = t, z, w, t \in D\}$ and let S_t be the fiber for fixed t . Note that when $t = 0$, S_t is a singular surface with a nodal point at $(z, w) = (0, 0)$, and when $t \neq 0$, S_t is smooth.

For $t \in D$ remove the discs $|z_i| < |t|$ from M_1 and M_2 and glue the remaining

surfaces by the annulus S_t according to the maps

$$z_1 \mapsto (z_1, \frac{t}{z_1}, t), \quad z_2 \mapsto (\frac{t}{z_2}, z_2, t).$$

This yields an analytic family $\mathcal{F} \longrightarrow D$ with fibres $M_t, t \neq 0$, being a compact Riemann surface of genus $g_1 + g_2$, and M_0 a stable Riemann surface (or curve) with node p (corresponding to p_1, p_2), i.e., M_0 is a point in the boundary of the Deligne-Mumford compactification of the moduli space \mathcal{M}_g .

We next choose symplectic homology bases of M_1 and M_2 to get a symplectic homology basis for M_t in such a way that $\{A_j, B_j \mid 1 \leq j \leq g_1\}$ are closed curves in $M_1 \cap M_t$ and similarly $\{A_j, B_j \mid g_1 < j \leq g_1 + g_2\}$ are closed curves in $M_2 \cap M_t$. For the proofs of the following proposition and its corollary see [12, p. 38] and [12, Corollary 3.2].

Proposition 3.1 *For sufficiently small t there is a normalized basis of abelian differentials $\omega_1, \dots, \omega_g$ for M_t , holomorphic in t , with the following expansions for $1 \leq i \leq g_1$ and $g_1 < j \leq g_1 + g_2$:*

$$\begin{aligned} \omega_i(x, t) &= \begin{cases} \omega_i^{(1)}(x) + O(t^2) & \text{for } x \in M_1 \setminus U_1, \\ -t\omega_i^{(1)}(p)\omega^{(2)}(x, p) + O(t^2) & \text{for } x \in M_2 \setminus U_2, \end{cases} \\ \omega_j(x, t) &= \begin{cases} \omega_j^{(2)}(x) + O(t^2) & \text{for } x \in M_2 \setminus U_2, \\ -t\omega_j^{(2)}(p)\omega^{(1)}(x, p) + O(t^2) & \text{for } x \in M_1 \setminus U_1, \end{cases} \end{aligned}$$

where the $\omega_i^{(1)}$ form a normalized basis for the abelian differentials on M_1 , the $\omega_j^{(2)}$ form a normalized basis for the abelian differentials on M_2 and $\omega^{(1)}(x, y), \omega^{(2)}(x, y)$ are the canonical differentials of the second kind on M_1, M_2 , which have poles (of order 2) only along the diagonal $x = y$.

Corollary 3.2 *The period matrix $\Pi(t)$ of M_t associated to the homology basis described above satisfies*

$$\lim_{t \rightarrow 0} \Pi(t) = \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{pmatrix}$$

where Π_1 (resp. Π_2) is the period matrix of M_1 (resp. M_2) with respect to their original homology bases.

Remark 3.3 As explained to the first author by R.Hain, this corollary also follows from general results of Schmid on degenerations and limits of variations of Hodge structures (see [17, p. 125]).

We now turn to the degeneration of a compact Riemann surface of genus g to a singular surface of genus $g - 1$ with a single, *non-separating* node. The construction is similar to that of separating nodes, except that we now glue in the annulus via local coordinates z_a and z_b centered at two distinct points a, b on a compact Riemann surface M of genus $g - 1$. Again the resulting surfaces form an analytic family $\mathcal{F} \longrightarrow D$ with fibres $M_t, t \neq 0$, each being a compact Riemann surface of genus g and M_0 a stable Riemann surface. The node is the identification of a and b in M_0 and does not disconnect the surface when removed (in contrast to the case considered above).

We choose a symplectic homology basis $\{A_j, B_j \mid 1 \leq j \leq g - 1\}$ for M away from the points a, b . The surfaces $M_t, t \neq 0$, each have genus g so we need two more loops A_g, B_g for a homology basis of these surfaces: take A_g as the boundary of the disk U_b , and B_g to run across the handle. One then has the following proposition; for the proof see [12, Corollary 3.8]. See also [29, Prop. 4.1].

Proposition 3.4 *For sufficiently small t the period matrix of M_t has the following expansion:*

$$\Pi(t) = \begin{pmatrix} \Pi_{ij} + t\pi_{ij} & a_i + t\pi_{ig-1} \\ a_j + t\pi_{g-1j} & -\frac{i}{2\pi} \log t + c_0 + c_1 t \end{pmatrix} + O(t^2)$$

where $\Pi = (\Pi_{ij})$ is the period matrix of M , $\lim_{t \rightarrow 0} \frac{O(t^2)}{t^2}$ is a finite matrix, and $a_j = \int_a^b \omega_j$.

4 The asymptotic cone of the Jacobian locus

Before proving the main Theorem 1.1 we emphasize the following fact.

Lemma 4.1 *The set \mathcal{D}_g of diagonal matrices in \mathcal{H}_g is a totally geodesic submanifold isometric to a product of g copies of the Poincaré hyperbolic plane \mathcal{H}_1 .*

Proof. Consider the map

$$\Phi : \prod_{k=1}^g Sp(1, \mathbb{R}) \longrightarrow Sp(g, \mathbb{R})$$

given by

$$\Phi\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}\right) := \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

where $D_1, D_2, D_3, D_4 \in \mathbb{C}^{g \times g}$ are *diagonal* matrices with entries $(a_1, \dots, a_g), (b_1, \dots, b_g), (c_1, \dots, c_g), (d_1, \dots, d_g)$, respectively. Clearly, Φ is an isomorphism of $\prod_{k=1}^g Sp(1, \mathbb{R})$ onto its image in $Sp(g, \mathbb{R})$. A direct calculation then shows that the orbit of $iI_g \in \mathcal{H}_g$ under $\Phi(\prod_{k=1}^g Sp(1, \mathbb{R}))$ is the set \mathcal{D}_g of all diagonal matrices in \mathcal{H}_g and, moreover, is isometric to the product of g real hyperbolic planes $\mathcal{H}_1 \cong Sp(1, \mathbb{R})/U(1) \cong SL_2(\mathbb{R})/SO(2)$. That this embedding is totally geodesic follows for instance from the Lie triple criterion (see [20], IV.7). \square

4.1 The proof of Theorem 1.1

Given a Riemann surface $M \in \mathcal{M}_g$, there are $g - 1$ separating curves such the corresponding singular surface is the union of g tori with punctures. Each torus coincides with its own Jacobian and corresponds to a point $z_k \in \mathcal{H}_1$ (resp. $\mathcal{A}_1 = Sp(1, \mathbb{Z}) \backslash \mathcal{H}_1$) for $k = 1, \dots, g$. We choose a homology basis for each torus as above and simultaneously shrink all $g - 1$ separating curves. Corollary 3.2 then implies that there exist period matrices $\Pi(t) \in \mathcal{H}_g$ of compact Riemann surfaces in \mathcal{M}_g such that

$$\lim_{t \rightarrow 0} \Pi(t) = \begin{pmatrix} z_1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & z_g \end{pmatrix} \in \mathcal{D}_g \subset \mathcal{H}_g$$

for the given $z_k \in \mathcal{H}_1$ for $k = 1, \dots, g$. Thus every point in the totally geodesic submanifold \mathcal{D}_g (Lemma 4.1) is the limit point of a sequence of period matrices of surfaces in \mathcal{M}_g in \mathcal{H}_g . In order to get the corresponding points in the period locus \mathcal{J}_g we have to pass to the quotient $\mathcal{A}_g = Sp(g, \mathbb{Z}) \backslash \mathcal{H}_g$ and in general there will be identifications. However, by Proposition 2.2, there are no identifications if we restrict to points in the Weyl chamber $\mathcal{C}_{a^*} \subset \mathcal{D}_g$ (consisting of certain *real*, positive diagonal matrices). We thus conclude that, given $\delta_1 > 0$ sufficiently small, the following holds: for any point $p \in \pi(\mathcal{C}_{a^*})$ there is a point $p' \in \mathcal{J}_g$ with $d_{\mathcal{A}_g}(p, p') < \delta_1$, i.e.

$$\pi(\mathcal{C}_{a^*}) \subset \mathcal{U}_{\delta_1}(\mathcal{J}_g).$$

On the other hand, also by Proposition 2.2, $\pi(\mathcal{C}_{a^*}) \subset \mathcal{A}_g$ is a net in the modular variety, i.e. there exists $\delta_2 > 0$ such that

$$\mathcal{J}_g \subset \mathcal{A}_g \subset \mathcal{U}_{\delta_2}(\pi(\mathcal{C}_{a^*})).$$

The constant $\delta_g := \delta_1 + \delta_2$ only depends on g . It follows that the Hausdorff distance between \mathcal{J}_g and $\pi(\mathcal{C}_{a^*})$ is finite:

$$d_{\mathcal{H}}(\mathcal{J}_g, \pi(\mathcal{C}_{a^*})) \leq \max\{\delta_1, \delta_2\}.$$

Consequently, with respect to the rescaled metrics $\frac{1}{n}d_{\mathcal{A}_g}$ the Hausdorff distance between \mathcal{J}_g and $\pi(\mathcal{C}_{a^*})$ goes to zero if $n \rightarrow \infty$. Finally, by Proposition 2.3, the asymptotic cone of \mathcal{A}_g is isometric to the Euclidean cone $\pi(\mathcal{C}_{a^*})$. The above estimates thus imply the claim of Theorem 1.1 and the proof is complete.

4.2 The proof of Theorem 1.2

For each elliptic curve C we fix an origin. Then there exists an involution ι such that the origin is a fixed point of ι . There is another fixed point of ι . The key point is to observe that for g elliptic curves C_1, \dots, C_g with such fixed involutions, if we glue them together in a chain along points of involution, then we get a stable hyperelliptic curve M_0 . We can open up these nodes of M_0 as in Section 3 while preserving an involution to get a smooth hyperelliptic curve M_t , i.e., we get a family of curves M_t in \mathcal{HE}_g which degenerates to M_0 . As in the proof of Theorem 1.1 one then shows that the totally geodesic submanifold \mathcal{D}_g of \mathcal{H}_g is contained in the closure of $J(\mathcal{HE}_g)$, and the same arguments as above complete the proof of Theorem 1.2.

5 Compactifications of Siegel modular varieties

Before we prove Theorem 1.3 and Corollary 1.4 we briefly review the Baily-Borel and the Borel-Serre compactification of moduli space \mathcal{A}_g .

5.1 The Baily-Borel compactification

First, we describe $\overline{\mathcal{A}}_g^{BB}$. Let $G = Sp(g, \mathbb{R})$ be the symplectic group with the split \mathbb{Q} -structure. For each maximal \mathbb{Q} -parabolic subgroup P of G , there is a Baily-Borel \mathbb{Q} -boundary component of \mathcal{H}_g , denoted by $e^{BB}(P)$, which is a Siegel upper half space of lower dimension and constructed as follows.

Let $P = N_P A_P M_P$ be the Langlands decomposition of P with respect to the maximal compact subgroup $U(g)$ of G . Here N_P is the unipotent radical of P , A_P is the split component, and M_P is a semisimple Lie group, and furthermore, A_P and M_P are stable under the Cartan involution associated with $U(g)$. Then

$$X_P = M_P / (U(g) \cap M_P)$$

is called the *boundary symmetric space* associated with the parabolic subgroup P . It turns out that X_P splits canonically as a product:

$$X_P = X_{h,P} \times X_{\ell,P},$$

where $X_{h,P}$ is a Hermitian symmetric space, and $X_{\ell,P}$ is a homothety section of a symmetric cone and thus also called a linear symmetric space. The Baily-Borel boundary component associated to the parabolic P is defined by

$$e^{BB}(P) = X_{h,P}.$$

The Baily-Borel compactification $\overline{\mathcal{A}}_g^{BB}$ is then constructed in two steps:

1. For every maximal proper \mathbb{Q} -parabolic subgroup P of $Sp(g, \mathbb{R})$, attach the \mathbb{Q} -boundary component $e^{BB}(P)$ to get a partial compactification

$$\overline{\mathcal{H}}_g^{BB} = \mathcal{H}_g \cup \coprod_{\text{maximal } \mathbb{Q}\text{-parabolic subgroups } P} e^{BB}(P).$$

2. Show that $\Gamma = Sp(g, \mathbb{Z})$ acts continuously on $\overline{\mathcal{H}}_g^{BB}$ with a compact quotient, which can be given the structure of a projective variety.

For example, for every $1 \leq k \leq g-1$, the \mathbb{Q} -boundary component $e^{BB}(P_{k,\infty})$ of \mathcal{H}_g that corresponds to the maximal \mathbb{Q} -parabolic subgroup

$$P_{k,\infty} = \left\{ \begin{pmatrix} A & 0 & B & n \\ m^t & u & n^t & b \\ C & 0 & D & -m \\ 0 & 0 & 0 & (u^{-1})^t \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(k, \mathbb{R}), u \in GL(g-k, \mathbb{R}) \right\}$$

$$m, n \in M_{k \times g-k}(\mathbb{R}), \quad b \in M_{g-k \times g-k}(\mathbb{R})$$

can be identified with the Siegel upper half space \mathcal{H}_k .

The topology of the partial compactification $\overline{\mathcal{H}}_g^{BB}$ is given by describing how sequences of interior points converge to boundary points. The boundary component $e^{BB}(P_{k,\infty}) \cong \mathcal{H}_k$ is attached at the infinity of \mathcal{H}_g as

$$\left\{ \begin{pmatrix} Z & 0 \\ 0 & +i\infty \end{pmatrix} \mid Z \in \mathcal{H}_k \right\} \cong \mathcal{H}_k,$$

where a sequence of points $Z_n \in \mathcal{H}_g$ converges to a point $Z_\infty \in \mathcal{H}_k$ if and only if when Z_n is written in the form $Z_n = \begin{pmatrix} Z'_n & Z'''_n \\ (Z'''_n)^t & Z''_n \end{pmatrix}$, where $Z'_n \in \mathcal{H}_k$, the following conditions are satisfied:

1. $Z'_n \rightarrow Z_\infty$ in \mathcal{H}_k ,
2. $\text{Im} Z''_n - (\text{Im} Z'''_n)^t (\text{Im} Z''_n)^{-1} (\text{Im} Z'''_n) \rightarrow +\infty$. (Note that for a sequence of real symmetric matrix $y_n \in \mathbb{C}^{n \times n}$, $y_n \rightarrow +\infty$ means that for every positive definite symmetric matrix $A \in \mathbb{C}^{n \times n}$, we have $y_n - A > 0$ when $n \gg 1$.)

The action of $Sp(n, \mathbb{Z})$ on \mathcal{H}_g extends to a continuous action on $\overline{\mathcal{H}}_g^{BB}$. By the reduction theory for $Sp(g, \mathbb{Z})$, it can be shown that every \mathbb{Q} -boundary component is a translate under $Sp(n, \mathbb{Z})$ of one of the $e^{BB}(P_{k, \infty}) = \mathcal{H}_k$ described above.

Proposition 5.1 *The Baily-Borel compactification $\overline{\mathcal{A}}_g^{BB}$ admits the following disjoint decomposition:*

$$\overline{\mathcal{A}}_g^{BB} = \mathcal{A}_g \cup \coprod_{k=0}^{g-1} \mathcal{A}_k,$$

where \mathcal{A}_0 consists of only one point.

Outline of the proof. To prove this result, the crucial point is to observe that though the induced action of $Sp(n, \mathbb{Z})$ on $\overline{\mathcal{H}}_g^{BB}$ is not properly discontinuous, for each \mathbb{Q} -boundary component $e^{BB}(P)$, it “effectively” induces a discrete action on it. Specifically, for the boundary component $e^{BB}(P_{k, \infty}) \cong \mathcal{H}_k$, two boundary points belong to one orbit of $Sp(g, \mathbb{Z})$ if and only if they belong to one orbit of the natural action of $Sp(k, \mathbb{Z})$ on \mathcal{H}_k .

By the reduction theory for $Sp(g, \mathbb{Z})$ (compare Proposition 2.1), the parabolic subgroups $P_{k, \infty}$, $k = 0, \dots, g-1$, are representatives of $Sp(g, \mathbb{Z})$ -conjugacy classes of proper \mathbb{Q} -parabolic subgroups of $Sp(n, \mathbb{R})$. Combined with the previous paragraph, it implies

$$Sp(g, \mathbb{Z}) \backslash \overline{\mathcal{H}}_g^{BB} = Sp(g, \mathbb{Z}) \backslash \mathcal{H}_g \cup \coprod_{k=0}^{g-1} Sp(k, \mathbb{Z}) \backslash \mathcal{H}_k = \mathcal{A}_g \cup \coprod_{k=0}^{g-1} \mathcal{A}_k,$$

which completes the proof. \square

For more details of the boundary components and the topology of $\overline{\mathcal{H}}_g^{BB}$ and $\overline{\mathcal{A}}_g^{BB}$ we refer to [26, pp. 36–37].

5.2 The Borel-Serre compactification

The Borel-Serre compactification $\overline{\mathcal{A}}_g^{BS}$ can be constructed as follows. For *every* proper \mathbb{Q} -parabolic subgroup P of $Sp(g, \mathbb{R})$, whether it is maximal or not, define its boundary component $e^{BS}(P)$ by

$$e^{BS}(P) = N_P \times X_P = N_P \times X_{h,P} \times X_{\ell,P}.$$

We emphasize that for every proper \mathbb{Q} -parabolic subgroup P of $G = Sp(g, \mathbb{R})$, whether it is maximal or not, there is a boundary symmetric space X_P , which also splits as the

product $X_P = X_{h,P} \times X_{\ell,P}$, and the Hermitian factor $X_{h,P}$ agrees with the Hermitian factor of a unique maximal \mathbb{Q} -parabolic subgroup P_{\max} containing P .

Then $\overline{\mathcal{A}}_g^{BS}$ is constructed in two steps:

1. For every \mathbb{Q} -parabolic subgroup P of $Sp(g, \mathbb{R})$, attach the \mathbb{Q} -boundary component $e^{BS}(P)$ to get a partial compactification

$$\overline{\mathcal{H}}_g^{BS} = \mathcal{H}_g \cup \coprod_{\mathbb{Q}\text{-parabolic subgroups } P} e^{BS}(P).$$

2. Show that $\Gamma = Sp(g, \mathbb{Z})$ acts continuously and properly on $\overline{\mathcal{H}}_g^{BS}$ with a compact quotient.

For every maximal \mathbb{Q} -parabolic subgroup P of $Sp(g, \mathbb{R})$, there is clearly a projection from the Borel-Serre boundary component $e^{BS}(P)$ to the Baily-Borel boundary component $e^{BB}(P)$. Similarly, for a non-maximal \mathbb{Q} -parabolic subgroup P , there is also a projection from $e^{BS}(P)$ to $e^{BB}(P_{\max})$. This suggests the following result (see [7] for the proof).

Proposition 5.2 *The identity map on \mathcal{A}_g extends to a continuous, surjective map from $\overline{\mathcal{A}}_g^{BS}$ to $\overline{\mathcal{A}}_g^{BB}$.*

6 The boundary points at infinity of \mathcal{J}_g in compactifications of \mathcal{A}_g

By Theorem 1.1 the Jacobian locus \mathcal{J}_g is “asymptotically dense”, e.g. in the sense that it forms a net in \mathcal{A}_g . In contrast Theorem 1.3 asserts that the boundary of the Jacobian locus \mathcal{J}_k is “small” in the boundary of the Baily-Borel compactification $\overline{\mathcal{A}}_g^{BB}$.

In order to prove Theorem 1.3, we need extension of the Jacobian map \overline{J} as stated in Proposition 1.5 in the introduction. We then determine the image of the boundary $\overline{\mathcal{M}}_g^{DM} - \mathcal{M}_g$ in $\overline{\mathcal{A}}_g^{BB}$ under \overline{J} .

6.1 The proof of Proposition 1.5

The moduli space \mathcal{M}_g is an orbifold and is covered by the Teichmüller space \mathcal{T}_g , which is a simply connected complex manifold. The Jacobian map $J : \mathcal{M}_g \rightarrow \mathcal{A}_g$ can be lifted to a map $\mathcal{T}_g \rightarrow \mathcal{H}_g$. Since the boundary of $\overline{\mathcal{M}}_g^{DM}$ consists of divisors with normal crossing, the Borel extension theorem in [6] yields the desired extension \overline{J} of the Jacobian map J .

6.2 The proof of Theorem 1.3.

First recall [18, p. 50] [27] that the boundary $\overline{\mathcal{M}}_g^{DM} - \mathcal{M}_g$ consists of $[\frac{g}{2}] + 1$ divisors, $D_0, D_1, \dots, D_{[\frac{g}{2}]}$, where a generic point of D_0 is a Riemann surface of genus $g - 1$ with two punctures, and for $k = 1, \dots, [\frac{g}{2}]$, a generic point of D_k is the union of a Riemann surface of genus k with one puncture and another Riemann surface of genus $g - k$ with one puncture.

Next note that for each $1 \leq k \leq [\frac{g}{2}]$, $\mathcal{H}_k \times \mathcal{H}_{g-k}$ is canonically embedded into \mathcal{H}_g by

$$(Z_1, Z_2) \in \mathcal{H}_k \times \mathcal{H}_{g-k} \rightarrow \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \in \mathcal{H}_g.$$

Similarly, the product $\mathcal{A}_{g_1} \times \mathcal{A}_{g_2}$ and hence the product of the Jacobian loci $\mathcal{J}_{g_1} \times \mathcal{J}_{g_2}$ are also mapped into \mathcal{A}_g by finite-to-one maps. In fact, if $g_1 \neq g_2$, $\mathcal{A}_{g_1} \times \mathcal{A}_{g_2}$ is embedded into \mathcal{A}_g . On the other hand, if $g_1 = g_2$, then the quotient of $\mathcal{A}_{g_1} \times \mathcal{A}_{g_2}$ by $\mathbb{Z}/2$ -action $(x, y) \mapsto (y, x)$ is embedded into \mathcal{A}_g . More generally, for every proper partition of g : $g_1 + \dots + g_j = g$, where $g_1, \dots, g_j \geq 1$, and $j \geq 2$, the product $\mathcal{H}_{g_1} \times \dots \times \mathcal{H}_{g_j}$ can be embedded into \mathcal{H}_g , and the product $\mathcal{J}_{g_1} \times \dots \times \mathcal{J}_{g_j}$ is also canonically mapped into \mathcal{A}_g by a finite-to-one map. We denote the image of $\mathcal{J}_{g_1} \times \dots \times \mathcal{J}_{g_j}$ in \mathcal{A}_g by $\mathcal{J}_{g_1} \times \dots \times \mathcal{J}_{g_j} / \sim$.

Recall from §3 that for every generic surface $M_0 \in D_k$, $k = 0, \dots, [\frac{g}{2}]$, there is an analytic family of Riemann surface M_t , $t \in D$, such that for $t \neq 0$, M_t is compact.

Suppose that $k \geq 1$. Then Corollary 3.2 (resp. its obvious generalization) implies that the limit $\lim_{t \rightarrow 0} J(M_t)$ exists and by the above remarks lies in \mathcal{A}_g . In particular, the generic points of $\overline{J}(D_k)$ are not contained in the boundary $\partial \overline{\mathcal{A}}_g^{BB}$.

Now assume that $k = 0$, and let \overline{M}_0 be the compact Riemann surface obtained from M_0 by adding the two punctures. Then the genus of \overline{M}_0 is equal to $g - 1$. Let $\overline{J}(\overline{M}_0)$ be the Jacobian of \overline{M}_0 , which is a point in \mathcal{A}_{g-1} . Identify \mathcal{A}_{g-1} with a subset of the boundary of $\overline{\mathcal{A}}_g^{BB}$ as in Proposition 5.1. Then $\overline{J}(\overline{M}_0)$ canonically determines a boundary point of $\overline{\mathcal{A}}_g^{BB}$.

By Proposition 3.4, the periods of the corresponding analytic family of surfaces M_t are given by

$$\Pi(t) = \begin{pmatrix} \Pi_{ij} + t\pi_{ij} & a_i + t\pi_{ig-1} \\ a_j + t\pi_{g-1j} & -\frac{i}{2\pi} \log t + c_0 + c_1 t \end{pmatrix} + O(t^2).$$

Note that $\Pi_{ij} + t\pi_{ij} \in \mathcal{A}_{g-1}$ converges to a point in \mathcal{J}_{g-1} , $a_i + t\pi_{ig-1}$ is bounded, and $\text{Im}(-\frac{i}{2\pi} \log t + c_0 + c_1 t) \rightarrow +\infty$ as $t \rightarrow 0$. By the definition of the convergence of interior points of \mathcal{A}_g to the boundary points of $\overline{\mathcal{A}}_g^{BB}$ in Section 5.1, we see that

the limit $\lim_{t \rightarrow 0} J(M_t)$ exists and is equal to the boundary point $J(\overline{M_0})$. This implies that when \mathcal{A}_{g-1} and hence \mathcal{J}_{g-1} is identified with a subset of the boundary $\partial \overline{\mathcal{A}_g}^{BB}$ as above, then the boundary $\partial \overline{\mathcal{J}_g}^{BB}$ of the Jacobian locus contains \mathcal{J}_{g-1} .

We need to show that the boundary $\partial \overline{\mathcal{J}_g}^{BB}$ is equal to the closure of \mathcal{J}_{g-1} and determine its closure. By Proposition 1.5, it is contained in the image of the boundary divisors $D_0, D_1, \dots, D_{[\frac{g}{2}]}$ under the extended map $\overline{J} : \overline{\mathcal{M}_g}^{DM} \rightarrow \overline{\mathcal{A}_g}^{BB}$. The above discussions show that it is contained in $\overline{J}(D_0)$. Let D'_0 be the open subvariety parametrizing Riemann surfaces of genus $g-1$ with two punctures. Then the above discussion shows that $\overline{J}(D'_0)$ is equal to \mathcal{J}_{g-1} and hence $\partial \overline{\mathcal{J}_g}^{BB}$ is equal to the closure of \mathcal{J}_{g-1} .

If $g \geq 5$, the closure of \mathcal{J}_{g-1} in \mathcal{A}_{g-1} is a proper subvariety (see [2] and the discussion in §1). This implies that the closure of \mathcal{J}_{g-1} in $\overline{\mathcal{J}_g}^{BB}$ is a proper subvariety of $\partial \overline{\mathcal{J}_g}^{BB}$. Note that the arguments in the previous paragraphs and the results in Propositions 3.1 and 3.4 can be generalized to the cases of multiple pinching geodesics and the closure of \mathcal{A}_{g-1} in the Baily-Borel compactification $\overline{\mathcal{A}_g}^{BB}$ is the Baily-Borel compactification of \mathcal{A}_{g-1} . Then by induction on g , we can show that the image $\overline{J}(D_0)$ and hence the boundary $\partial \overline{\mathcal{J}_g}^{BB}$ is equal to the union of \mathcal{J}_k , $k \leq g-1$. This completes the proof of the case when $g \geq 5$, i.e., the second statement of Theorem 1.3.

If $g \leq 4$, then \mathcal{J}_{g-1} is dense in \mathcal{A}_{g-1} . Since \mathcal{A}_{g-1} is dense in $\partial \overline{\mathcal{A}_g}^{BB}$, this implies that for $g \leq 4$, $\partial \overline{\mathcal{J}_g}^{BB}$ is equal to $\partial \overline{\mathcal{A}_g}^{BB}$, which proves the first part of Theorem 1.3.

6.3 The proof of Corollary 1.4.

Let $g = 2$ or 3 . Then \mathcal{J}_g is dense in \mathcal{A}_g . This clearly implies that $\partial \overline{\mathcal{J}_g}^{BS}$ is equal to $\partial \overline{\mathcal{A}_g}^{BS}$. By Proposition 5.2, the boundary $\partial \overline{\mathcal{J}_g}^{BS}$ is contained in the inverse image under the map $\overline{\mathcal{J}_g}^{BS} \rightarrow \overline{\mathcal{J}_g}^{BB}$ of the boundary $\partial \overline{\mathcal{J}_g}^{BB}$.

Now let $g \geq 5$. By Theorem 1.3, $\overline{\mathcal{J}_g}^{BB}$ is a proper closed subset of $\partial \overline{\mathcal{J}_g}^{BB}$. It follows that for $g \geq 5$, $\partial \overline{\mathcal{J}_g}^{BS}$ is a proper subset of $\partial \overline{\mathcal{A}_g}^{BS}$ of strictly smaller dimension.

Remark 6.1 The above arguments miss the case $g = 4$. It is natural to conjecture that in this case, $\partial \overline{\mathcal{J}_g}^{BS}$ is also a proper subset of $\partial \overline{\mathcal{A}_g}^{BS}$ of strictly smaller dimension.

7 The interior boundary points of \mathcal{J}_g in \mathcal{A}_g

In the previous section we determined the boundary points of the Jacobian locus \mathcal{J}_g at infinity of the Siegel modular variety \mathcal{A}_g . In this section we study the closely related

problem of identifying the interior boundary points of \mathcal{J}_g in \mathcal{A}_g . More precisely, let $\overline{\mathcal{J}}_g$ be the closure of \mathcal{J}_g in \mathcal{A}_g , and let $\partial\mathcal{J}_g = \overline{\mathcal{J}}_g - \mathcal{J}_g$.

Recall that an abelian variety is *irreducible* if it is not isomorphic to a product of two abelian varieties of smaller dimensions. Then the following result on Jacobian varieties is well-known.

Proposition 7.1 *For every compact Riemann surface $M \in \mathcal{M}_g$, its Jacobian $J(M)$ is an irreducible principally polarized abelian variety.*

We recall the ideas of the proof for convenience. By [14, p. 320], every principally polarized abelian variety has a Riemann theta-divisor. As mentioned in the introduction, the Jacobian variety $J(M)$ is canonically a principally polarized abelian variety. Denote its theta-divisor by Θ . By a theorem of Riemann [14, p. 338], Θ is equal to a translate of the image W_{g-1} of the symmetric power M^{g-1} . Since M and hence M^{g-1} is irreducible, Θ is irreducible.

On the other hand, if $J(M)$ is a reducible principally polarized abelian variety, $J(M) \cong A_1 \times A_2$, and Θ_1, Θ_2 are the Riemann theta-divisors of A_1, A_2 , then $\Theta = \Theta_1 \times A_2 + A_1 \times \Theta_2$ and is reducible. This contradiction proves Proposition 7.1.

Proposition 7.1 implies that the sets of reducible Jacobians $\mathcal{J}_{g_1} \times \cdots \times \mathcal{J}_{g_j} / \sim$ are disjoint from the Jacobian locus \mathcal{J}_g .

Proposition 7.2 *The closure $\overline{\mathcal{J}}_g$ of \mathcal{J}_g in \mathcal{A}_g admits the following decomposition:*

$$\overline{\mathcal{J}}_g = \mathcal{J}_g \cup \coprod_{g_1 + \cdots + g_j = g, j \geq 2} \mathcal{J}_{g_1} \times \cdots \times \mathcal{J}_{g_j} / \sim .$$

Therefore, the interior boundary $\partial\mathcal{J}_g = \overline{\mathcal{J}}_g \setminus \mathcal{J}_g$ is given by

$$\partial\mathcal{J}_g = \coprod_{g_1 + \cdots + g_j = g, j \geq 2} \mathcal{J}_{g_1} \times \cdots \times \mathcal{J}_{g_j} / \sim .$$

Proof. It is well-known that a sequence of Riemann surfaces M_n in \mathcal{M}_g converges to a stable Riemann surface in the boundary of $\overline{\mathcal{M}}_g^{DM}$ if and only if a collection of simple, disjoint closed geodesics on the Riemann surfaces M_n are pinched. Proposition 3.4 (resp. a straight forward generalization) and the discussion in the previous section imply that if there is a non-separating geodesic, then the images $J(M_n)$ diverge to infinity in \mathcal{A}_g .

Therefore, we can assume that all pinching geodesics are separating. Then these pinching geodesics determine a partition of g : $g_1 + \cdots + g_j = g$. By Corollary 3.2

(resp. its direct generalization) and the discussion in Section 6.2, their corresponding periods $J(M_n)$ converge to a point in the subset $\mathcal{J}_{g_1} \times \cdots \mathcal{J}_{g_j} / \sim$ of reducible Jacobians, and every point in $\mathcal{J}_{g_1} \times \cdots \mathcal{J}_{g_j} / \sim$ is the limit of such a degenerating sequence. This proves that $\partial\mathcal{J}_g$ contains the union of $\mathcal{J}_{g_1} \times \cdots \mathcal{J}_{g_j} / \sim$ for every proper partition $g = g_1 + \cdots + g_j$.

In order to show that $\partial\mathcal{J}_g$ is actually *equal* to this union, we need the extension result in Proposition 1.5. In terms of the boundary divisors D_i of $\overline{\mathcal{M}}_g^{DM}$ and the extended period map $\overline{J} : \overline{\mathcal{M}}_g^{DM} \rightarrow \overline{\mathcal{A}}_g^{BB}$ in Proposition 1.5, the boundary $\partial\mathcal{J}_g$ is contained in the union $\bigcup_{k=0}^{\lfloor \frac{g}{2} \rfloor} \overline{J}(D_k) \cap \mathcal{A}_g$. As in the proof of Theorem 1.3, let D'_0 be the open subvariety parametrizing Riemann surfaces of genus $g-1$ with two punctures. Then the arguments there imply that $\overline{J}(D'_0)$ is not contained in \mathcal{A}_g . This implies that the boundary $\partial\mathcal{J}_g$ is contained in the union $\bigcup_{k=1}^{\lfloor \frac{g}{2} \rfloor} \overline{J}(D_k) \cap \mathcal{A}_g$.

As in the proof of Theorem 1.3 again, Propositions 3.1 and 3.4 and their generalizations to the case of multiple pinching curves imply that every point in $\bigcup_{k=1}^{\lfloor \frac{g}{2} \rfloor} \overline{J}(D_k) \cap \mathcal{A}_g$ is equal to the Jacobian of the stable curve obtained by pinching only separating simple, disjoint closed curves. (The point is that whenever a non-separating curve of Riemann surfaces is pinched, their Jacobian varieties will go to the boundary of \mathcal{A}_g .) Then by induction on g , it can be shown that $\bigcup_{k=1}^{\lfloor \frac{g}{2} \rfloor} \overline{J}(D_k) \cap \mathcal{A}_g$ is equal to the union $\bigcup_{g_1 + \cdots + g_j, j \geq 2} \mathcal{J}_{g_1} \times \cdots \mathcal{J}_{g_j} / \sim$. This completes the proof of Proposition 7.2.

Remark 7.3 The interior boundary set $\partial\mathcal{J}_g$ has already been identified in [25, p. 74], though without a detailed proof. Related results are also hinted in [16]. We emphasize that without Proposition 1.5, one can only conclude that $\partial\mathcal{J}_g$ contains the union $\bigsqcup_{g_1 + \cdots + g_j, j \geq 2} \mathcal{J}_{g_1} \times \cdots \mathcal{J}_{g_j} / \sim$.

As pointed out above, \mathcal{A}_g is a Zariski open subset of the normal projective variety $\overline{\mathcal{A}}_g^{BB}$. By [2], the closure of the image $J(\mathcal{M}_g)$ in $\overline{\mathcal{A}}_g^{BB}$ with respect to the regular topology is an algebraic subvariety, i.e., the image $J(\mathcal{M}_g)$ is a quasi-projective variety. Since $\dim \mathcal{M}_g = 3g - 3$ and $\dim \mathcal{A}_g = g(g+1)/2$, it follows that for $g = 2, 3$ and only for these values of g , $J(\mathcal{M}_g)$ is Zariski dense in \mathcal{A}_g .

Corollary 7.4 *When $g = 2, 3$, the complement of \mathcal{J}_g in \mathcal{A}_g consists exactly of the reducible principally polarized abelian varieties.*

Proof. When $g = 2, 3$, \mathcal{J}_g is Zariski dense in \mathcal{A}_g . Then the complement of \mathcal{J}_g in \mathcal{A}_g is equal to the interior boundary $\partial\mathcal{J}_g$, which consists of reducible Jacobians by Proposition 7.2. For every proper partition $g_1 + \cdots + g_j = g$ with $g = 2, 3$ and $g_1, \dots, g_j \leq 2$, it follows that every reducible principally polarized abelian variety of

dimension g is a reducible Jacobian of a stable curve. This completes the proof of Corollary 7.4.

8 Remarks on the distortion of \mathcal{J}_g inside \mathcal{A}_g

Recall that for any path connected subspace B of a geodesic metric space (A, d_A) , there is an induced length function d_B on B . Then a function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is called a *distortion function* for B inside A if for all pairs of points of $p, q \in B$,

$$d_B(p, q) \leq f(d_A(p, q)).$$

Clearly, if B is a totally geodesic subspace, then one can take $f(x) = x$.

In [13, Problem 4.12], Farb raised the following problem.

Compute the distortion of the Jacobian locus \mathcal{J}_g inside \mathcal{A}_g .

The expectation in [13] is that any distortion function of \mathcal{J}_g in \mathcal{A}_g with respect to the locally symmetric metric is huge and might be exponential. Based on the results of the previous sections, we think that the distortion of \mathcal{J}_g in \mathcal{A}_g is of a quite different nature. In fact, we suspect that there are sequences of pairs of points $p_n, q_n \in \mathcal{J}_g$ such that $d_{\mathcal{J}_g}(p_n, q_n)$ is bounded away from 0, or even goes to infinity, while $d_{\mathcal{A}_g}(p_n, q_n)$ goes to 0. Roughly speaking, when \mathcal{M}_g is embedded into \mathcal{A}_g , it is folded up, and for some parts near the boundary of \mathcal{M}_g , the different sheets are becoming closer and closer.

To provide such examples of sequences of Riemann surfaces in \mathcal{M}_g , we consider the case $g = 4$ and start with four distinct Riemann surfaces of genus 1: S_1, S_2, S_3, S_4 . Fix them for the moment, though they will move to infinity of \mathcal{M}_1 later. Glue them together to get analytic families as in §3 in the order S_1, S_2, S_3, S_4 . In particular, S_1 is only connected to S_2 , but S_2 is glued to both S_1 and S_3 . Similarly, S_3 is glued with both S_2 and S_4 , and S_4 is only glued with S_3 . Such a family of Riemann surfaces depends on three parameters $t_1, t_2, t_3 \in D$. We denote these surfaces by M_{t_1, t_2, t_3} .

Now we switch the order and glue the surfaces together in the order S_2, S_1, S_4, S_3 so that S_1 is now glued with both S_2 and S_4 , but S_2 is only glued with S_1 . We denote this new family by $\tilde{M}_{t_1, t_2, t_3}$. Due to the different orders, the compact Riemann surfaces M_{t_1, t_2, t_3} and $\tilde{M}_{t_1, t_2, t_3}$ in \mathcal{M}_4 are not isomorphic to each other for t_1, t_2, t_3 sufficiently small.

Now under the extended Jacobian map, the degenerate surfaces $M_{0,0,0}$ and $\tilde{M}_{0,0,0}$ have the same image:

$$\overline{J}(M_{0,0,0}) = \overline{J}(\tilde{M}_{0,0,0}).$$

The crucial point is that these images do not depend on the order of S_1, S_2, S_3, S_4 . In fact, they are both equal to the product of the Jacobians $J(S_1), J(S_2), J(S_3), J(S_4)$. This means that when t_1, t_2, t_3 are very small, $d_{\mathcal{A}_g}(J(M_{t_1, t_2, t_3}), J(\tilde{M}_{t_1, t_2, t_3}))$ is small. (Note that they are in the interior of \mathcal{A}_g and are close to each other.)

On the other hand, in view of Proposition 3.1, when $t_1, t_2, t_3 \rightarrow 0$, the images $J(M_{t_1, t_2, t_3})$ are basically contained in a Weyl chamber of \mathcal{A}_g . Similarly, for the other family $\tilde{M}_{t_1, t_2, t_3}$, the images $J(\tilde{M}_{t_1, t_2, t_3})$ are basically contained in another Weyl chamber. This implies that $d_{\mathcal{J}_g}(J(M_{t_1, t_2, t_3}), J(\tilde{M}_{t_1, t_2, t_3}))$ is bounded away from 0. In fact, it is likely that the distance $d_{\mathcal{J}_g}(J(M_{t_1, t_2, t_3}), J(\tilde{M}_{t_1, t_2, t_3}))$ goes to infinity. The reason is that in order to go from one such chamber to another one through the Jacobian locus \mathcal{J}_g , there is no shortcut, and we need to go through a fixed compact region in \mathcal{A}_g (or \mathcal{M}_g). Then Proposition 3.1 implies the claimed growth of the distance.

The above discussion indicates that one can cut \mathcal{M}_g into finitely many suitable pieces, whose images in \mathcal{A}_g under the Jacobian map J have a distortion that is asymptotically negligible.

References

- [1] W. BAILY, Satake's compactification of V_n , *Amer. J. Math.* **80** (1958), 348–364.
- [2] W. BAILY, On the theory of θ -functions, the moduli of abelian varieties, and the moduli of curves, *Ann. of Math.* **75** (1962) 342–381.
- [3] W. BAILY, A. BOREL, Compactification of arithmetic quotients of bounded symmetric domains, *Ann. of Math.* **84** (1966), 442–528.
- [4] A. BEAUVILLE, Le problème de Schottky et la conjecture de Novikov, Séminaire Bourbaki 1986/87, *Astérisque* **152-153** (1988), 101–112.
- [5] A. BOREL, Introduction aux groupes arithmétiques, Paris, 1969.
- [6] A. BOREL, Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem, *J. Differential Geometry* **6** (1972), 543–560.
- [7] A. BOREL, L. JI, Compactifications of symmetric and locally symmetric spaces, *Mathematics: Theory & Applications*, Birkhäuser Boston, 2006.
- [8] A. BOREL, J.P. SERRE, Corners and arithmetic groups, *Comment. Math. Helv.* **48** (1973), 436–491.

- [9] P. BUSER, P. SARNAK, On the period matrix of a Riemann surface of large genus (with an appendix by J.H. Conway and N.J.A. Sloane), *Invent. math.* **117** (1994), 27–56.
- [10] O. DEBARRE, The Schottky problem: an update, *Current topics in complex algebraic geometry, Math. Sci. Res. Inst. Publ.* **28**, Cambridge, 1995, 57–64.
- [11] P. DELIGNE, D. MUMFORD, The irreducibility of the space of curves of given genus, *Inst. Hautes Études Sci. Publ. Math.* **36** (1969), 75–109.
- [12] J.D. FAY, Theta Functions on Riemann surfaces, *Lecture Notes in Mathematics* **352**, Springer, 1973.
- [13] B. FARB, Some problems on mapping class groups and moduli space, In: Problems on mapping class groups and related topics, pp. 11–55, *Proc. Sympos. Pure Math.* **74**, Amer. Math. Soc., Providence, 2006.
- [14] P. GRIFFITHS, J. HARRIS, Principles of algebraic geometry, John Wiley & Sons, Inc., New York, 1994. xiv+813 pp.
- [15] M. GROMOV, Metric structures for Riemannian and Non-Riemannian spaces, *Progress in Mathematics* **152**, Birkhäuser, 1999.
- [16] R. HAIN, Locally symmetric families of curves and Jacobians, in *Moduli of curves and abelian varieties*, 91–108, Aspects Math., E33, Vieweg, Braunschweig, 1999.
- [17] R. HAIN, Periods of limit mixed Hodge structures, in *Current developments in mathematics*, 113–133, Int. Press, Somerville, MA, 2003.
- [18] J. HARRIS, I. MORRISON, Moduli of curves. *Graduate Texts in Mathematics*, **187**. Springer-Verlag, New York, 1998. xiv+366 pp.
- [19] T. HATTORI, Asymptotic geometry of arithmetic quotients of symmetric spaces, *Math. Z.* **222** (1996), 247–277.
- [20] S. HELGASON, Differential Geometry, Lie Groups, and Symmetric Spaces, *Pure and applied mathematics* **80**, Academic Press, 1978.
- [21] L. JI, R. MACPHERSON, Geometry of compactifications of locally symmetric spaces, *Ann. Inst. Fourier, Grenoble* **52** (2002), 457–559.
- [22] B. KLEINER, B. LEEB, Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings, *Inst. Hautes Études Sci. Publ. Math.* **86** (1997), 115–197.

- [23] E. LEUZINGER, Tits Geometry, Arithmetic Groups, and the Proof of a Conjecture of Siegel, *J. Lie Theory* **14** (2004), 317–338.
- [24] E. LEUZINGER, Reduction theory for mapping class groups and applications to moduli spaces, preprint 2008.
- [25] D. MUMFORD, Curves and their Jacobians, The University of Michigan Press, Ann Arbor, Mich., 1975. vi+104 pp.
- [26] Y. NAMIKAWA, Toroidal compactification of Siegel spaces, *Lecture Notes in Mathematics*, **812**. Springer, Berlin, 1980. viii+162 pp.
- [27] R. VAKIL, The moduli space of curves and its tautological ring, *Notices Amer. Math. Soc.* **50** (2003), no. 6, 647–658.
- [28] C.L. SIEGEL, Collected works, Vol. II, p.108
- [29] R. WENTWORTH, The Asymptotics of the Arakelov-Green’s function and Faltings’ Delta invariant, *Commun. Math. Phys.* **137** (1991), 427–459.

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